ELEN E3401: Electromagnetics

Spring 2025

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Lecture #10

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Vector fields analysis (review)

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

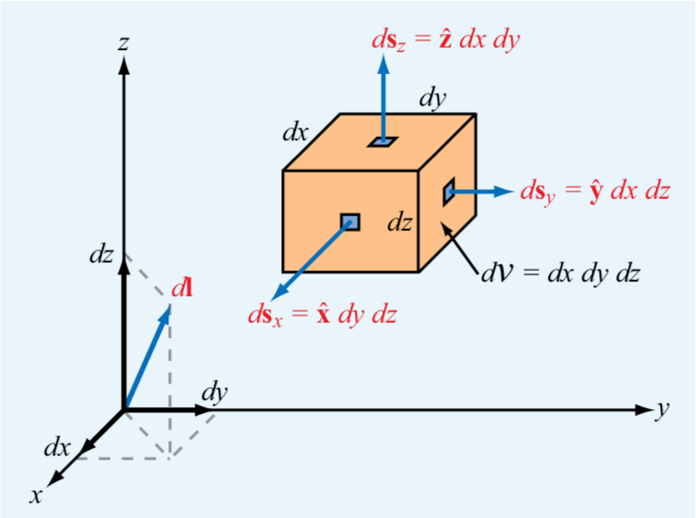
Cartesian Coordinates

Differential length vector: $d\vec{l} = \hat{x}dl_x + \hat{y}dl_y + \hat{z}dl_z = \hat{x}\mathrm{dx} + \hat{y}dy + \hat{z}dz$

Differential area vector: $d\vec{S}_x = \hat{x}dl_y dl_z = \hat{x}dydz$ (y-z plane)

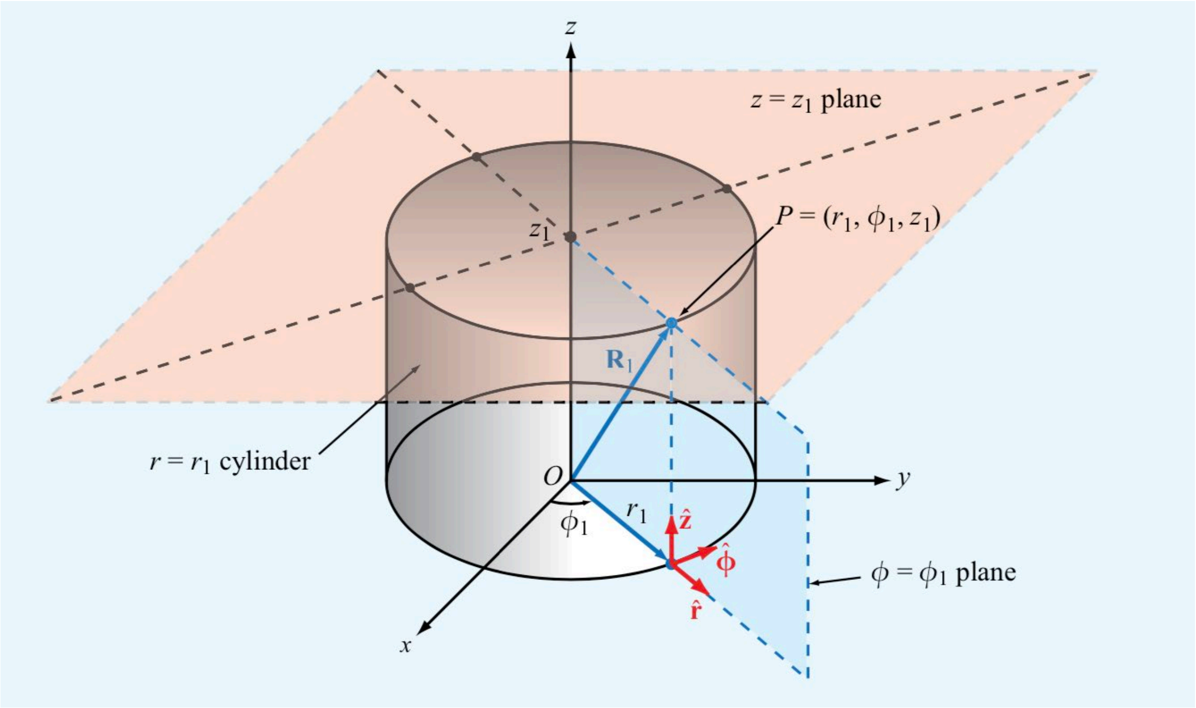
$$d\vec{S}_y = \hat{y}dxdz$$

$$d\vec{S}_z = \hat{z}dxdy$$



Differential volume: $dV = dxdydz$

Cylindrical Coordinates



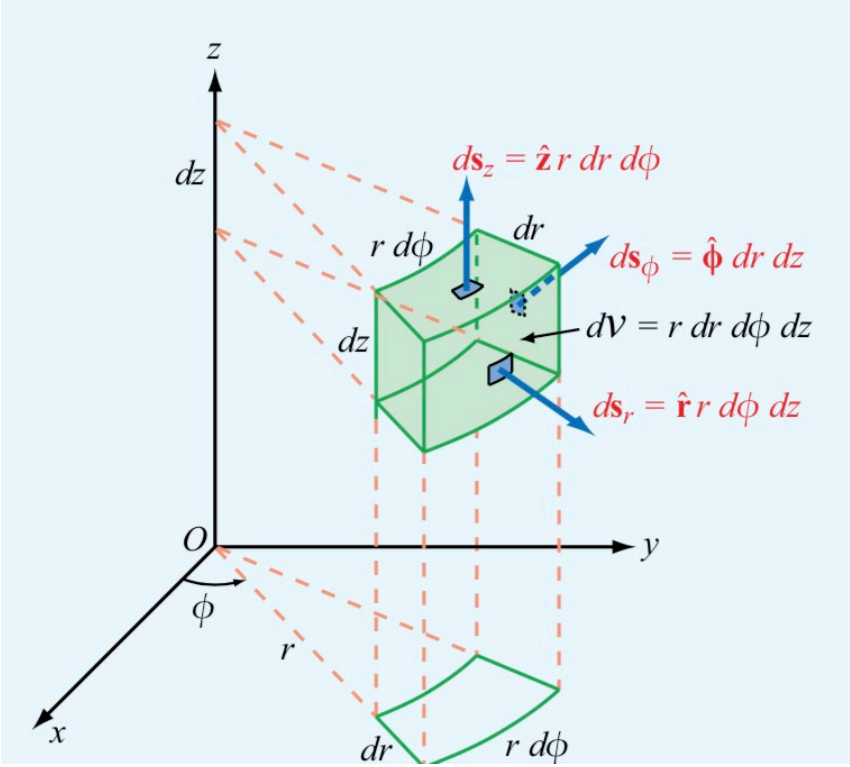
$0 \leq r < \infty \qquad 0 \leq \varphi < 2\pi \qquad -\infty < z < \infty$

$\hat{r} \times \hat{\varphi} = \hat{z} \quad \hat{\varphi} \times \hat{z} = \hat{r} \quad \hat{z} \times \hat{r} = \hat{\varphi}$

r → φ → z

$$\vec{A} = \hat{a}\left|\vec{A}\right| = \hat{r}A_r + \hat{\varphi}A_\varphi + \hat{z}A_z$$

Cylindrical Coordinates



Differential Length:

$$dl_r = dr, dl_\varphi = rd\varphi, dl_z = dz$$

$$d\vec{l} = \hat{r}dl_r + \hat{\varphi}dl_\varphi + \hat{z}dl_z$$

$$d\vec{l} = \hat{r}dr + \hat{\varphi}rd\varphi + \hat{z}dz$$

Differential surface area:

$$d\vec{S}_r = \hat{r}dl_\varphi dl_z = \hat{r}rd\varphi dz$$ ($\varphi - z$ cylindrical surface)

$$d\vec{S}_\varphi = \hat{\varphi}dl_r dl_z = \hat{\varphi}drdz$$ ($r - z$ plane)

$$d\vec{S}_z = \hat{z}dl_r dl_\varphi = \hat{z}rdrd\varphi$$ ($r - \varphi$ plane)

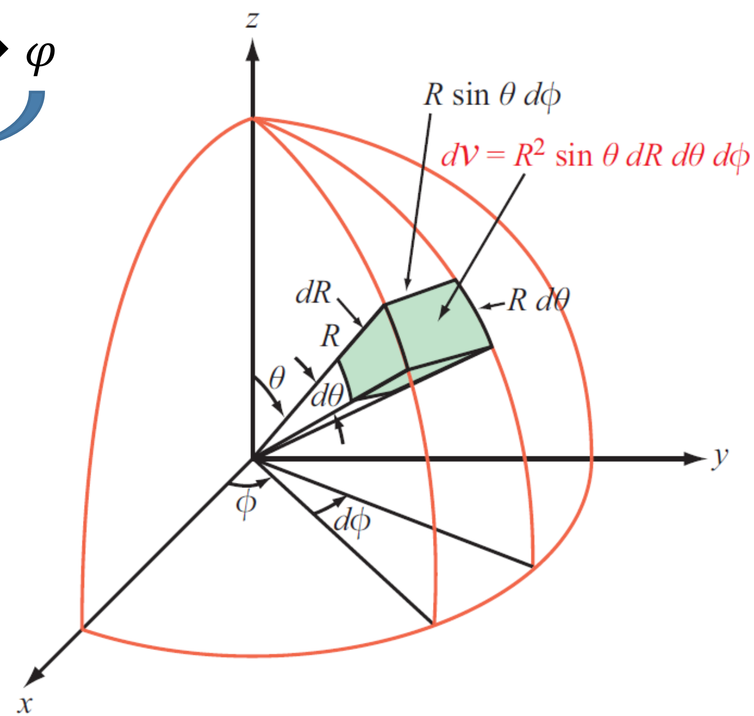
Differential volume: $d\mathcal{V} = dl_r dl_\varphi dl_z = r\, drd\varphi dz$

Spherical Coordinates

$\hat{R} \times \hat{\theta} = \hat{\varphi}$ $\quad \hat{\theta} \times \hat{\varphi} = \hat{R}$ $\quad \hat{\varphi} \times \hat{R} = \hat{\theta}$ $\quad R \rightarrow \theta \rightarrow \varphi$

$$\vec{A} = \hat{a}|\vec{A}| = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\varphi}A_\varphi$$

$$0 \leq R < \infty \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \varphi < 2\pi$$



<u>Differential Length:</u>

$$dl_R = dR, dl_\theta = Rd\theta, dl_\varphi = Rsin\theta d\varphi$$

$$d\vec{l} = \hat{R}dl_R + \hat{\theta}dl_\theta + \hat{\varphi}dl_\varphi$$

$$d\vec{l} = \hat{R}dR + \hat{\theta}Rd\theta + \hat{\varphi}Rsin\theta d\varphi$$

<u>Differential surface area:</u>

$$d\vec{S}_R = \hat{R}dl_\theta dl_\varphi = \hat{R}R^2 sin\theta d\theta d\varphi$$ ($\theta - \varphi$ spherical surface)

$$d\vec{S}_\theta = \hat{\theta}dl_R dl_\varphi = \hat{\theta}Rsin\theta dRd\varphi$$ (R $- \varphi$ plane)

$$d\vec{S}_\varphi = \hat{\varphi}dl_R dl_\theta = \hat{\varphi}RdRd\theta$$ ($R - \theta$ plane)

<u>Differential volume:</u> $d\mathcal{V} = dl_R dl_\theta dl_\varphi = R^2 sin\theta dRd\theta d\varphi$

Transformations

Cartesian to cylindrical:

$$r = {}^{+}\sqrt{x^2 + y^2} \qquad \varphi = tan^{-1}\left(\frac{y}{x}\right)$$

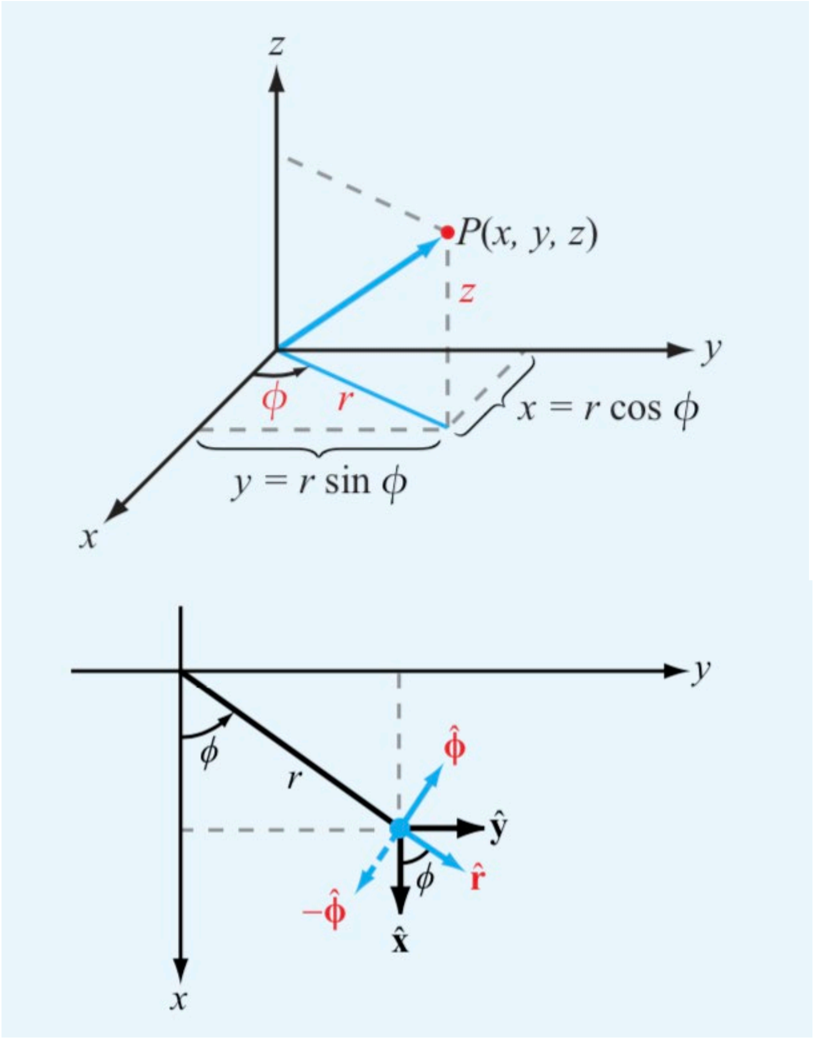
$$x = rcos\varphi \qquad y = rsin\varphi$$

$$\hat{r} \cdot \hat{x} = cos\varphi \qquad \hat{r} \cdot \hat{y} = sin\varphi$$

$$\hat{\varphi} \cdot \hat{x} = -sin\varphi \qquad \hat{\varphi} \cdot \hat{y} = cos\varphi$$

$$\begin{cases} \hat{r} = \hat{x}cos\varphi + \hat{y}sin\varphi \\ \hat{\varphi} = -\hat{x}sin\varphi + \hat{y}cos\varphi \end{cases}$$

$$\begin{cases} \hat{x} = \hat{r}cos\varphi - \hat{\varphi}sin\varphi \\ \hat{y} = \hat{r}sin\varphi + \hat{\varphi}cos\varphi \end{cases}$$



Transformations

<u>Cartesian to spherical:</u>

$$R = \sqrt[+]{x^2 + y^2 + z^2} \qquad \theta = tan^{-1}\left(\frac{\sqrt[+]{x^2 + y^2}}{z}\right) \qquad \varphi = tan^{-1}\left(\frac{y}{x}\right)$$

$$x = Rsin\theta cos\varphi \qquad y = Rsin\theta sin\varphi \qquad z = Rcos\theta$$

$$\begin{cases} \hat{R} = \hat{x}sin\theta cos\varphi + \hat{y}sin\theta sin\varphi + \hat{z}cos\theta \\ \hat{\theta} = \hat{x}cos\theta cos\varphi + \hat{y}cos\theta sin\varphi - \hat{z}sin\theta \\ \hat{\varphi} = -\hat{x}sin\varphi + \hat{y}cos\varphi \end{cases}$$

$$\begin{cases} \hat{x} = \hat{R}sin\theta cos\varphi + \hat{\theta}cos\theta cos\varphi - \hat{\varphi}sin\varphi \\ \hat{y} = \hat{R}sin\theta sin\varphi + \hat{\theta}cos\theta sin\varphi + \hat{\varphi}cos\varphi \\ \hat{z} = \hat{R}cos\theta - \hat{\theta}sin\theta \end{cases}$$

Gradient of Scalar Field

We will introduce: Gradient → for scalar fields   Divergence, curl → for vector fields

Example of gradient → scalar field → temperature   Vector Analysis

$\vec{E}$ and $\vec{H}$ are vector fields

Scalar Field → temperature $T(x, y, z)$

$$d\vec{l} = \hat{x}\mathrm{dx} + \hat{y}dy + \hat{z}dz$$

dT along differential direction $d\vec{l}$

$$dT = \underbrace{\left[\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right]}_{\text{Vector = temperature change in direction}} \cdot d\vec{l}$$

Gradient of T ($grad\ T$)

$$\vec{\nabla}T = grad\ T = \hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}$$

Define gradient operator: $\boxed{\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$

Gradient of Scalar Field

Gradient operator on scalar function → results in vector with:

mag = max rate of change of physical scalar per unit distance

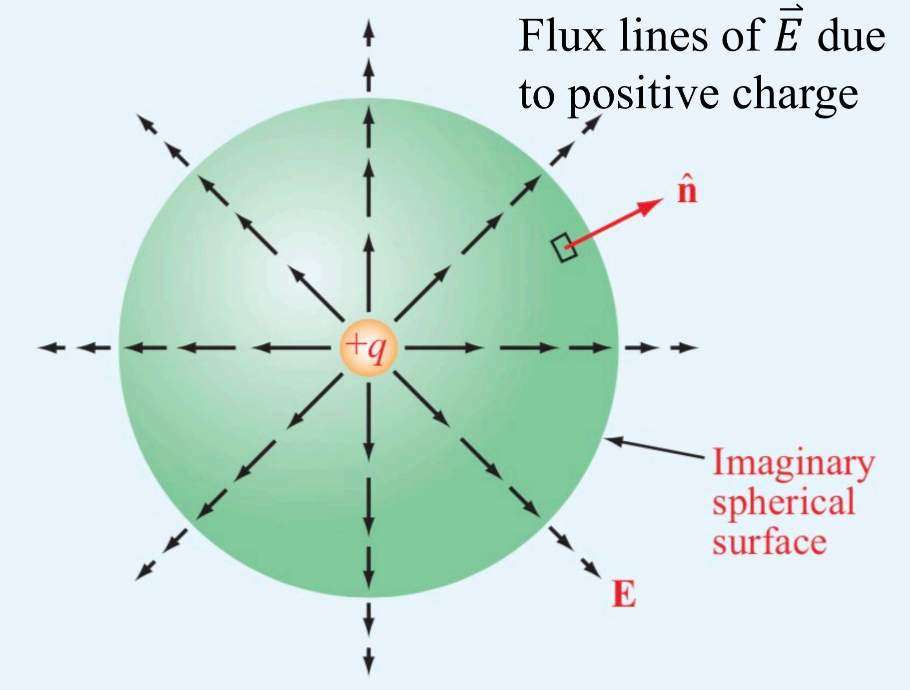
Direction = direction of max increase

$$\boxed{\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}}$$

$$\vec{\nabla}_{cyl} = \hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}$$

$$\vec{\nabla}_{sph} = \hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{Rsin\theta}\frac{\partial}{\partial \varphi}$$

Divergence of Vector Field



Recall Coulomb's Law → positive point charge q

$\vec{E}$ field will point outward, proportional to q and decreases as $\frac{1}{R^2}$

$$\vec{E} = \hat{R}\,\frac{q}{4\pi\epsilon_0 R^2}\ \text{(V/m)}$$

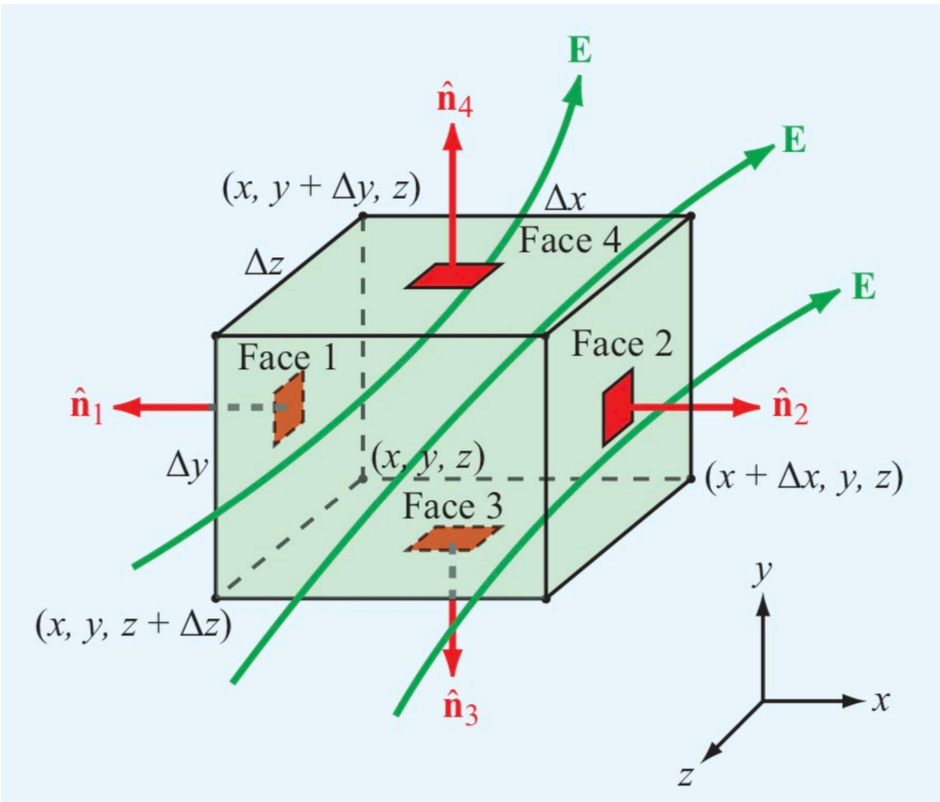
At surface boundary → define <u>flux density</u>: amount of outward flux crossing unit surface

<u>Flux density</u> of $\vec{E} = \frac{\vec{E}\cdot d\vec{s}}{|d\vec{s}|} = \frac{\vec{E}\cdot\hat{n}ds}{ds} = \vec{E}\cdot\hat{n}$, $\hat{n}$ = normal to $d\vec{s}$

Total flux $= \oint_S \vec{E}\cdot d\vec{s}$

Enclosed imaginary surface

Divergence of Vector Field



Differential rectangular volume for divergence of $\vec{E}$

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

We sum up the flux through the 6 faces:

<u>Face 1</u>

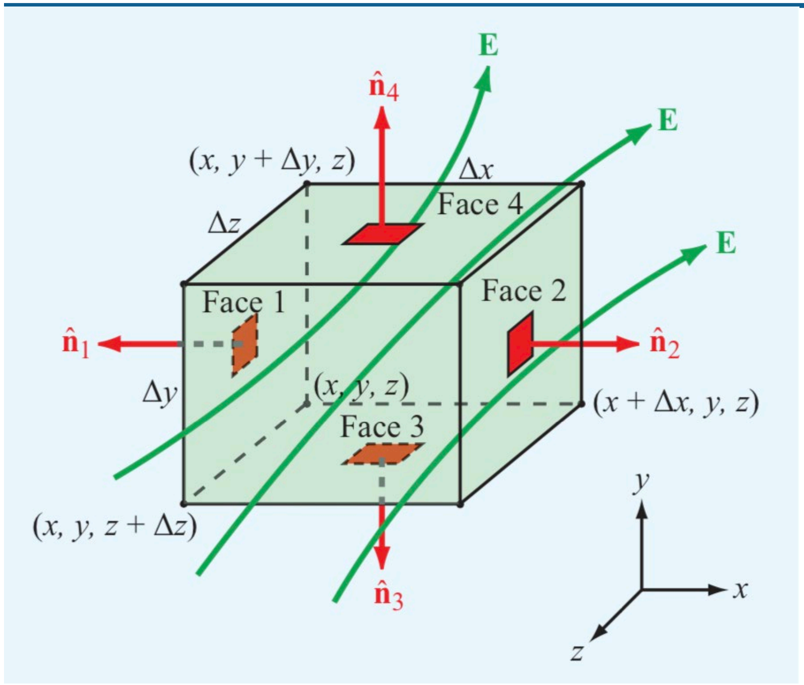
$$F_1 = \int_{face1} \vec{E} \cdot \hat{n}_1 ds$$

$$= \int (\hat{x}E_x + \hat{y}E_y + \hat{z}E_z) \cdot (-\hat{x}) dydz$$

$$F_1 = -E_x(1)\Delta y \Delta z$$

Value at center (small surface)

Divergence of Vector Field



Differential rectangular volume for divergence of $\vec{E}$

$$\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$$

Face 2

$$F_2 = E_x(2)\Delta y \Delta z$$

We shrink volume to Δx and have:

$E_x(2) = E_x(1) + \frac{\partial E_x}{\partial x}\Delta \text{x}$ (Taylor series 1st order)

$$F_2 = [E_x(1) + \frac{\partial E_x}{\partial x}\Delta \text{x}]\Delta y \Delta z$$

$$F_1 + F_2 = \frac{\partial E_x}{\partial x}\Delta \text{x}\Delta y \Delta z$$

Do same $\underbrace{F_3, F_4}_{\frac{\partial E_y}{\partial y}}, \underbrace{F_5, F_6}_{\frac{\partial E_z}{\partial z}}$

Divergence of Vector Field

$$\oint_S \vec{E} \cdot d\vec{s} = \left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] \Delta x \Delta y \Delta z$$

$$\oint_S \vec{E} \cdot d\vec{s} = (div\ \vec{E})\Delta\mathcal{V} \longleftarrow \text{Differential volume}$$

Shrink $\Delta\mathcal{V} \rightarrow 0$ div $\vec{E}$: divergence of $\vec{E}$ $\rightarrow$ div $\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

Shrink $\Delta\mathcal{V} \rightarrow 0$ $\rightarrow$ divergence of $\vec{E}$ defined at a point as (div $\vec{E}$): net outward flux per unit volume over closed surface:

$$\text{div}\ \vec{E} \equiv \lim_{\Delta V \to 0} \frac{\oint_S \vec{E} \cdot d\vec{s}}{\Delta\mathcal{V}}$$

S is surface encloses elemental volume $\mathcal{V}$

We use $\vec{\nabla} \cdot \vec{E}$ to indicate div $\vec{E}$

$$\boxed{\vec{\nabla} \cdot \vec{E} = div\ \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}}$$

$\vec{\nabla} \cdot \vec{E}$:

- positive if <u>net</u> flux lines <u>out</u> so enclosed volume contains <u>source</u>
- negative if <u>net</u> flux lines <u>in</u> so enclosed volume contains <u>sink</u>
- Zero: net flux =0 $\rightarrow$ divergence-less

Divergence of Vector Field

$$\oint_S \vec{E} \cdot d\vec{s} = \vec{\nabla} \cdot \vec{E} \Delta \mathcal{V} \rightarrow \int_v \vec{\nabla} \cdot \vec{E} d\mathcal{V}$$

Divergence theorem

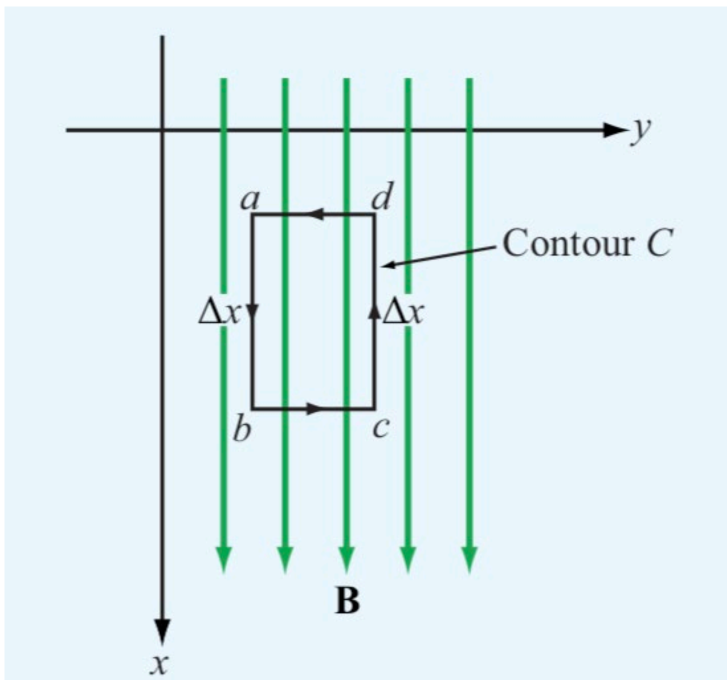
$$\underbrace{\int_v \vec{\nabla} \cdot \vec{E} d\mathcal{V}}_{\text{Divergence of field from volume}} = \underbrace{\oint_S \vec{E} \cdot d\vec{s}}_{\text{Sum of flux through surface enclosing the volume}}$$

Curl of Vector Field

Curl of vector, $\vec{B}$ → rotation or circulation

$$circulation = \oint_C \vec{B} \cdot d\vec{l}$$

Consider uniform $\vec{B}$ field: $\vec{B} = \hat{x}B_o$ magnetic flux density

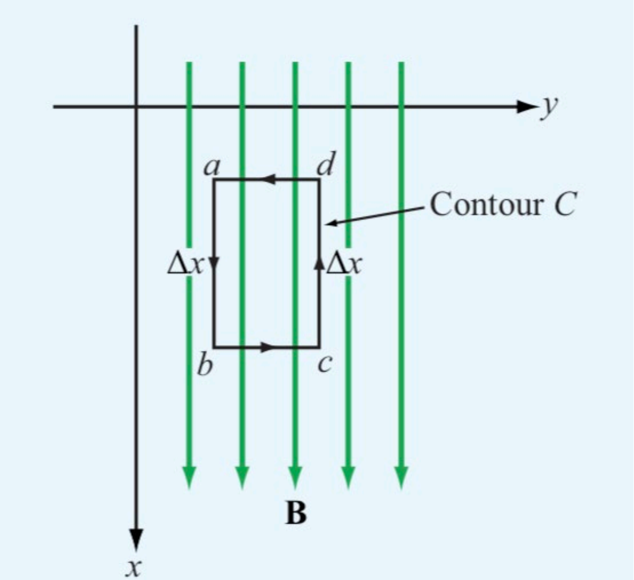


(a) Uniform field

Curl of Vector Field

Curl of vector, $\vec{B}$ → rotation or circulation: $\quad circulation = \oint_C \vec{B} \cdot d\vec{l}$

Uniform $\vec{B}$ field: $\vec{B} = \hat{x}B_o$ magnetic flux density



(a) Uniform field

$$circulation = \int_a^b \hat{x}B_o \cdot \hat{x}dx + \overbrace{\int_b^c \hat{x}B_o \cdot \hat{y}dy}^{=0 \text{ since } \hat{x}\cdot\hat{y}=0} + \int_c^d \hat{x}B_o \cdot \hat{x}dx + \underbrace{\int_d^a \hat{x}B_o \cdot \hat{y}dy}_{=0 \text{ since } \hat{x}\cdot\hat{y}=0}$$

$$circulation = B_o\Delta x - B_o\Delta x = 0$$

$$\Delta x = b - a = c - d$$

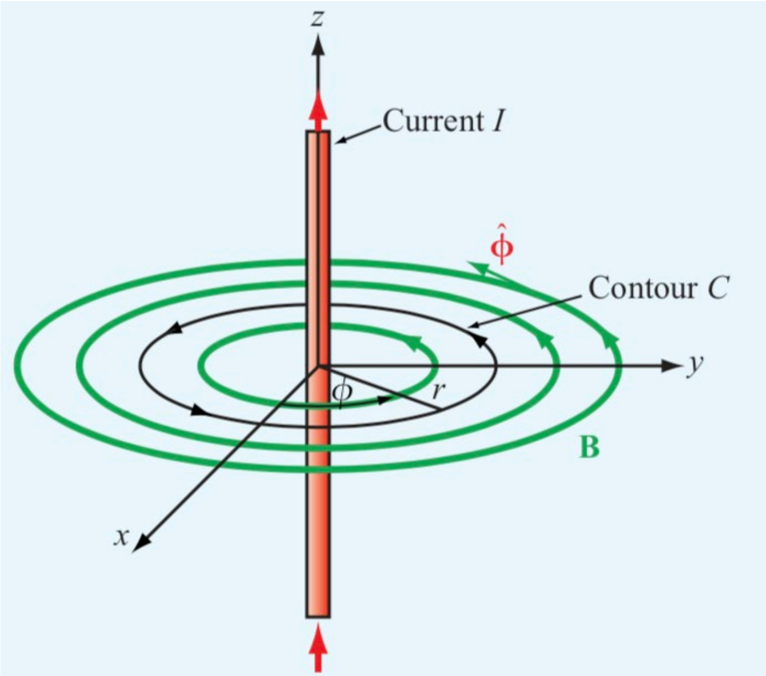
Circulation of <u>uniform</u> field = 0

Curl of Vector Field

Consider magnetic flux density, $\vec{B}$ of infinite wire with dc-current = I

$$\vec{B} = \hat{\varphi}\frac{\mu_0 I}{2\pi r}$$

For circular contour, C, at radius r in x-y plane: $d\vec{l} = \hat{\varphi} r d\varphi$



(b) Azimuthal field

$$circulation\ of\ \vec{B} = \int_C \vec{B}\cdot d\vec{l}$$

$$circulation = \int_0^{2\pi} \hat{\varphi}\frac{\mu_0 I}{2\pi r}\cdot\hat{\varphi} r d\varphi = \mu_0 I$$

Curl of Vector Field

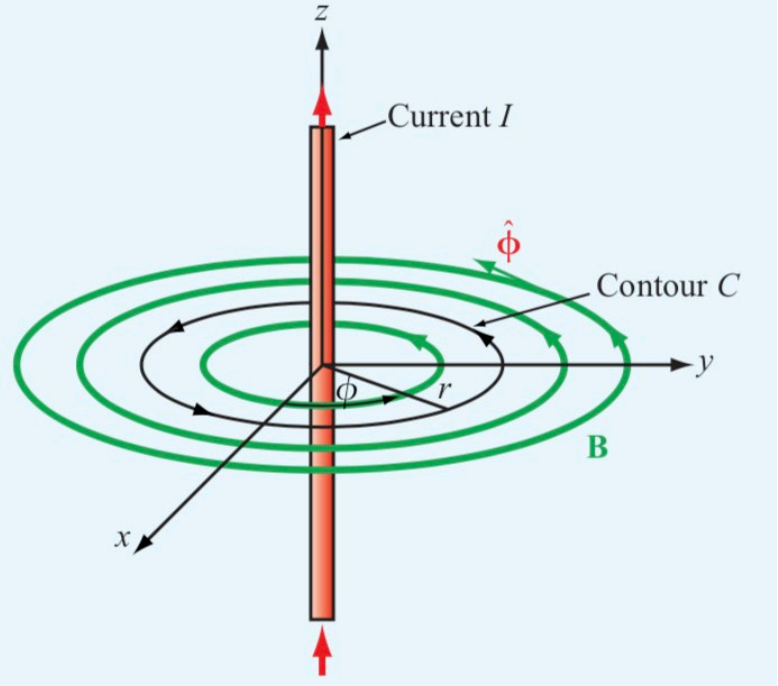
magnetic flux density, $\vec{B}$ of infinite wire with dc-current = I

$$\vec{B} = \hat{\varphi}\frac{\mu_0 I}{2\pi r}$$

$$circulation\ of\ \vec{B} = \int_c \vec{B}\cdot d\vec{l}$$

Take contour in x-z or y-z planes

circulation = 0 because there is no φ component



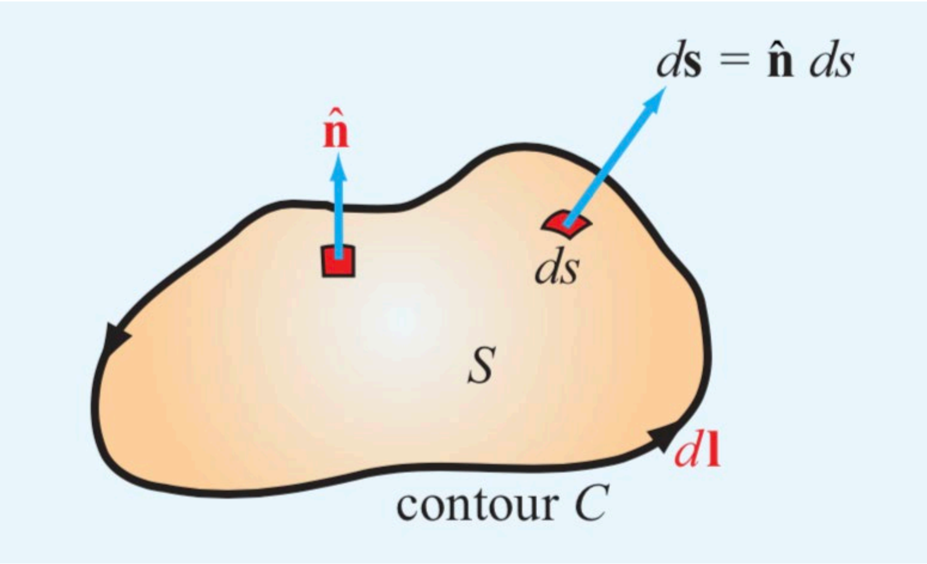
(b) Azimuthal field

→ Circulation depends on contour and direction

$$\vec{\nabla}\times\vec{B} = \lim_{\Delta s\to 0}\frac{1}{\Delta s}\left[\hat{n}\oint_c \vec{B}\cdot d\vec{l}\right]_{max}$$

Curl of Vector Field

$\vec{\nabla} \times \vec{B}$ = curl of $\vec{B}$ = circulation of $\vec{B}$ per unit area, with area ΔS of contour C selected such that circulation is maximum



$$\vec{B} = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z$$

$$\vec{\nabla} \times \vec{B} = \hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Useful Vector Identities

$$\vec{\nabla} \times \left(\vec{A} + \vec{B}\right) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$$

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$$

$$\vec{\nabla} \times \left(\vec{\nabla} V\right) = 0$$

Stoke's Theorem

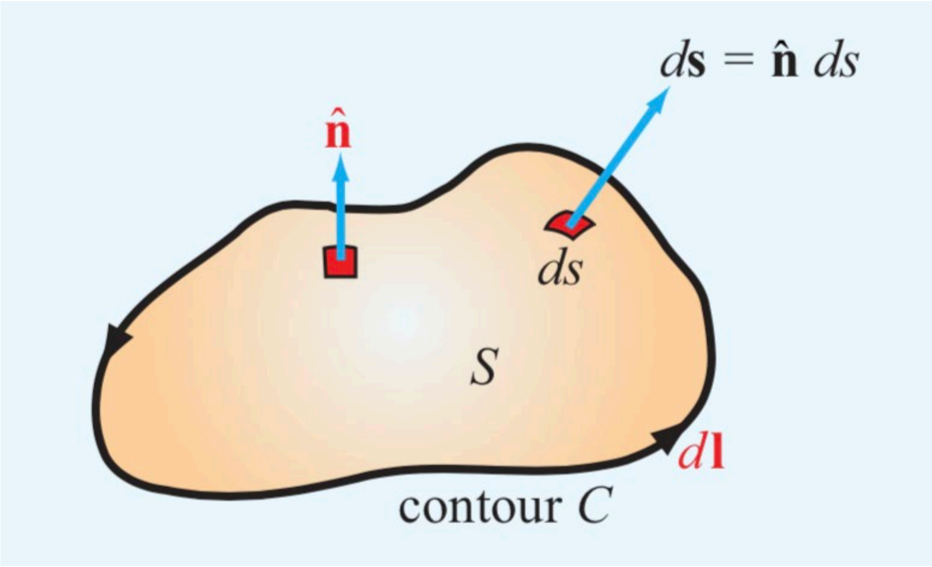
Surface integral of curl of vector over open surface *S*

→ equivalent to line integral of vector along contour, *C*, bounding surface, *S*

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \oint_C \vec{B} \cdot d\vec{l}$$

Surface integral of field curl

Contour line integral of field enclosing surface



Laplacian Operator

Divergence of the gradient of a scalar: $\vec{\nabla} \cdot (\vec{\nabla} V)$

$$\vec{\nabla} V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

$$\vec{\nabla} \cdot (\vec{\nabla} V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Laplacian of scalar: $\nabla^2 V = \vec{\nabla} \cdot (\vec{\nabla} V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

<u>Laplacian of vector:</u> $\vec{E} = \hat{x}E_x + \hat{y}E_y + \hat{z}E_z$

$$\nabla^2 \vec{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\vec{E}$$

$$\nabla^2 \vec{E} = \hat{x}\underbrace{\nabla^2 E_x}_{\text{Laplacian of vector component}} + \hat{y}\nabla^2 E_y + \hat{z}\nabla^2 E_z$$

$$\nabla^2 \vec{E} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla} \times (\vec{\nabla} \times \vec{E})$$

Maxwell’s Equations

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

- Hold in any material including vacuum

- 1873 James Clark Maxwell obtained from experiments

  by: Coulomb, Gauss, Ampere, Faraday

  Unified theory of electricity and magnetism

Maxwell's Equations

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$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left.\begin{array}{l}\vec{E} = \text{electric field intensity} \\ \vec{D} = \text{electric field flux density}\end{array}\right\} \quad \vec{D} = \epsilon \vec{E} \qquad \epsilon = \text{electrical permittivity}$$

$$\left.\begin{array}{l}\vec{H} = \text{magnetic field intensity} \\ \vec{B} = \text{magnetic flux density}\end{array}\right\} \quad \vec{B} = \mu \vec{H} \qquad \mu = \text{magnetic permeability}$$

ρ_V = electric charge density per unit volume

$\vec{J}$ = current density per unit area

Maxwell's Equations

Static → $\frac{\partial}{\partial t} = 0$, ρ_V and $\vec{J}$ are constant

Electrostatics:

$$\vec{\nabla} \cdot \vec{D} = \rho_V$$

$$\vec{\nabla} \times \vec{E} = 0$$

Static → decoupled

Magneto statics:

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} = \vec{J}$$

Electrostatics:

many applications – sensors, displays

Define sources – charge densities, current distributions

Maxwell's Equations

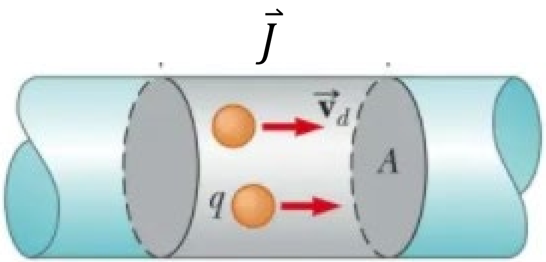
<u>Charge density</u> ρ_V = volume charge density C/m^3 = $\frac{dq}{d\mathcal{V}}$

$$Q = \int_{\mathcal{V}} \rho_V \, d\mathcal{V} \;\; [C]$$

ρ_s = surface charge density [C/m^2]

$$\rho_s = \frac{dq}{dS} \quad [\text{C/m}^2]$$

ρ_l = line charge density = $\frac{dq}{dl}$ [C/m]



<u>Current density</u>

$$\vec{J} = \rho_V \vec{u} \qquad [\text{A/m}^2]$$

Charge velocity

$$\vec{I} = \int_S \vec{J} \cdot d\vec{s} \qquad [\text{A}]$$

Total current flowing through surface, S

Maxwell's Equations

Coulomb's Law

1) Isolated charge q → induces $\vec{E}$ field: $\vec{E} = \hat{R}\frac{q}{4\pi\epsilon_0 R^2}$ (V/m) at point P

2) In presence of $\vec{E}$ field, force on test charge, q': $\vec{F} = q'\vec{E}$

(N) (C) (N/C = V/m)

$\vec{D} = \epsilon\,\vec{E}$ in material with electric permittivity, ϵ

$\epsilon = \epsilon_r\epsilon_0$ $\quad\quad$ $\epsilon_0 = 8.85 \times 10^{-12} \sim \frac{1}{36\pi} \times 10^{-9}$

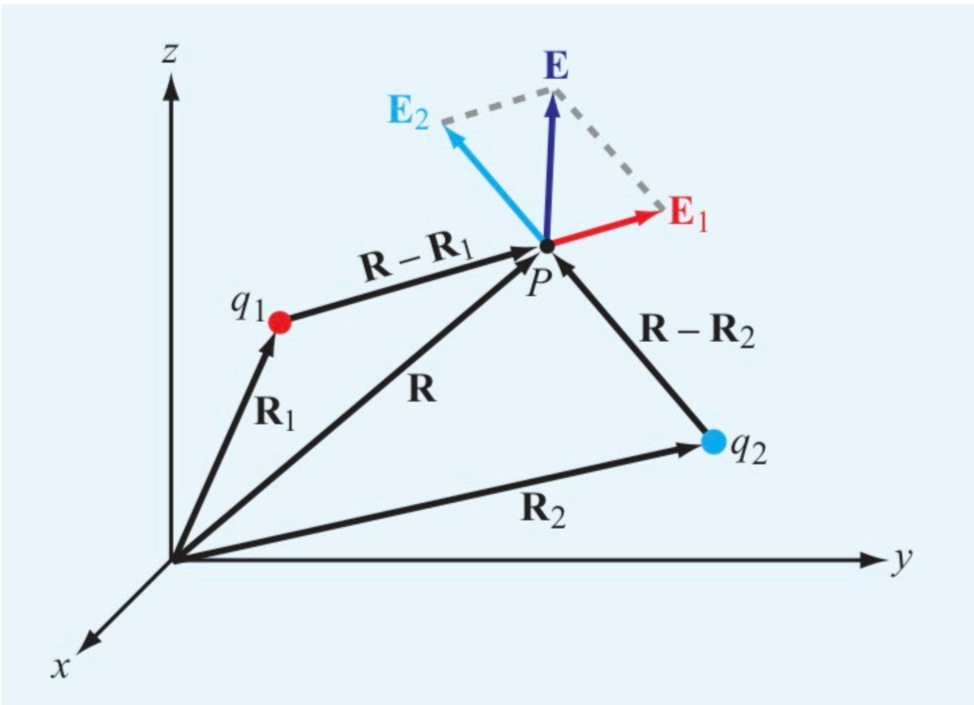
$\epsilon_r = \frac{\epsilon}{\epsilon_0}$ relative permittivity (dielectric constant)

ϵ: independent of magnitude of $\vec{E}$ → linear

independent of $\vec{E}$ direction → isotropic

Note: permittivity can be nonlinear, non-isotropic

Multiple Charges



$$\vec{E} = \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} \frac{q_i(\vec{R} - \vec{R}_i)}{|R - R_i|^3}$$

$$\vec{E} = \hat{R} \frac{q}{4\pi\epsilon R^2}$$

$$\frac{(\vec{R} - \vec{R}_1)}{|R - R_1|} \qquad \frac{1}{|R - R_1|^2}$$

Charge distributions

Consider differential = dq = $\rho_V d\mathcal{V}$

$$d\vec{E} = \hat{R}' \frac{dq}{4\pi\epsilon R'^2} = \hat{R}' \frac{\rho_V d\mathcal{V}'}{4\pi\epsilon R'^2}$$

Differential $\vec{E}$ due to differential charge dq

Volume distribution: $$\vec{E} = \int_{\mathcal{V}'} d\vec{E} = \frac{1}{4\pi\epsilon} \int_{\mathcal{V}'} \hat{R}' \frac{\rho_V d\mathcal{V}}{R'^2}$$

Surface distribution: $$\vec{E} = \frac{1}{4\pi\epsilon} \int_{S'} \hat{R}' \frac{\rho_s ds'}{R'^2}$$

Line distribution: $$\vec{E} = \frac{1}{4\pi\epsilon} \int_{l'} \hat{R}' \frac{\rho_l dl'}{R'^2}$$

Gauss's Law

Differential form of Gauss's Law

From Coulomb's Law → obtained expression for $\vec{E}$

$\vec{\nabla} \cdot \vec{D} = \rho_V$ The divergence of the electric flux density = charge density

Can convert to integral form: $\int_{\mathcal{V}} \vec{\nabla} \cdot \vec{D} d\mathcal{V} = \int_{\mathcal{V}} \rho_V d\mathcal{V} = Q$

Total enclosed charge in volume, $\mathcal{V}$

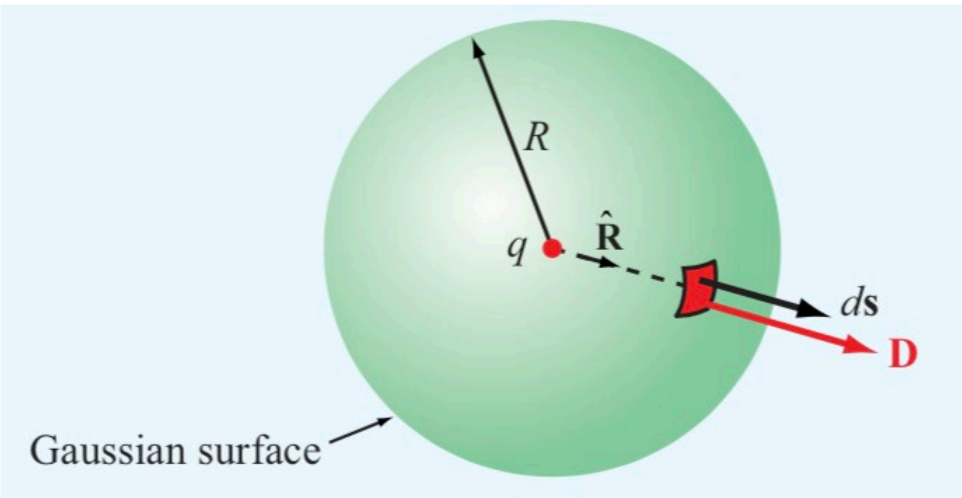
Divergence theorem: $\underbrace{\int_{v} \vec{\nabla} \cdot \vec{E} d\mathcal{V}}_{\text{Integral of field divergence from volume, } \mathcal{V}} = \underbrace{\oint_{S} \vec{E} \cdot d\vec{s}}_{\text{Net flux of field through closed surface that bounds volume, } \mathcal{V}}$

For $\vec{D}$:

Gauss's Law Integral $\left\{ \int_{\mathcal{V}} \vec{\nabla} \cdot \vec{D} d\mathcal{V} = \oint_{S} \vec{D} \cdot d\vec{s} = Q \right.$

Gauss's Law – obtain E field from point charge

Consider point charge, q, enclosed by spherical surface of arbitrary radius, R



$$\vec{D} = \hat{R} D_R$$

$$d\vec{s} = \hat{R} ds \qquad \int_0^{\pi}\int_0^{2\pi} R^2 \underbrace{sin\theta d\theta}_{2}\,\underbrace{d\varphi}_{2\pi}$$

Apply Gauss's Law:

$$\oint_S \vec{D} \cdot d\vec{s} = \oint_S \hat{R} D_R \cdot \hat{R} ds$$

$$= \oint_S D_R ds = D_R 4\pi R^2 = q$$

$$D_R 4\pi R^2 = q \qquad D_R = \frac{q}{4\pi R^2}$$

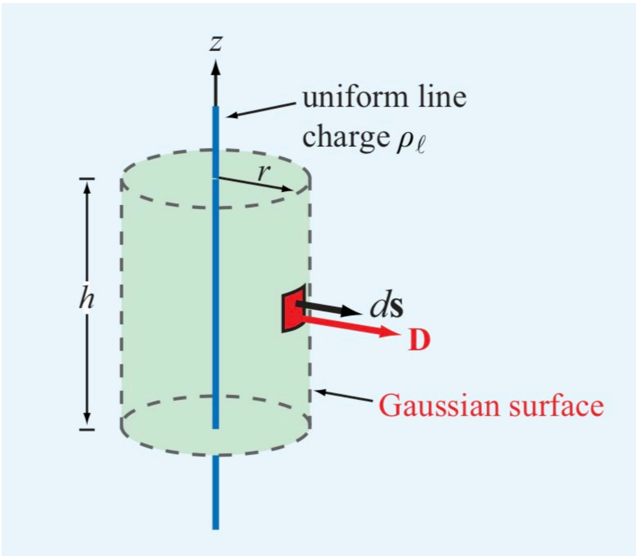
$$\vec{D} = \epsilon\, \vec{E} \qquad \vec{E} = \frac{\vec{D}}{\epsilon} = \hat{R}\frac{q}{4\pi\epsilon R^2} \ \text{(V/m)}$$ same as from Coulomb's Law

Gauss's Law easier to apply for <u>charge distribution</u>:

integrate $\vec{D}$ over <u>chosen Gaussian surface</u>

Choose Gaussian surface such that $\vec{D}$ is constant in magnitude and either normal or parallel (tangential) to surface

$\vec{E}$ field of infinite line charge



From symmetry, $\vec{D}$ in radial direction, $\hat{r}$. Cannot depend on φ or z.

$$\vec{D} = \hat{r}D_r$$

Choose cylinder Gaussian surface of radius, r and height, h – around line charge

Total charge within cylinder → Q = $\rho_l\ h$

Since $\vec{D}$ is along $\hat{r}$, top and bottom of cylinder do not contribute (no field lines emerging)

Only curved cylindrical surface:

$$\oint_S \vec{D} \cdot d\vec{s} = Q$$

$$\int_{z=0}^{h} \int_{\varphi=0}^{2\pi} \hat{r}D_r \cdot \hat{r}rd\varphi dz = \rho_l\ h$$

$$2\pi \mathrm{h} D_r \mathrm{r} = \rho_l\ h$$

Infinite line charge: $\vec{E} = \dfrac{\vec{D}}{\epsilon_0} = \hat{r}\dfrac{D_r}{\epsilon_0} = \hat{r}\dfrac{\rho_l}{2\pi\epsilon_0 r}$